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General Dynamical Equations for Dingle's Space-Times Filled with a Charged Non-perfect Fluid

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Abstract In this paper we have assumed charged non-perfect fluid as the material content of the space-time. The expression for the "*mass function*-$M(r, y, z, t)$" is obtained for the general situation and the contributions from the Ricci tensor in the form of material energy density ρ, pressure anisotropy $[\frac{p_2+p_3}{2} - p_1]$, electromagnetic field energy $\mathcal{E}$ and the conformal Weyl tensor, viz. energy density of the free gravitational field ϵ $(= \frac{-3\Psi_2}{4\pi})$ are made explicit. This work is an extension of the work obtained earlier by Rao and Hasmani (Math. Today XIIA:71, 1993; New Directions in Relativity and Cosmology, Hadronic Press, Nonantum, 1997) for deriving general dynamical equations for Dingle's space-times described by this most general orthogonal metric,

$$ds^2 = \exp(\nu)dt^2 - \exp(\lambda)dr^2 - \exp(2\alpha)dy^2 - \exp(2\beta)dz^2,$$

where ν, λ, α and β are functions of all four space-time variables r, y, z and t.

Keywords Newman-Penrose formalism · General dynamical equations · Energy-momentum scalar · Free gravitational energy

1 Introduction

The general dynamical equations for spherically symmetric space-times were given by Lemaitre [5], Bondi [1], Misner and Sharp [7], Podurets (1964) and Ruban (1983). The formalism is based on the definition of the "*mass function*-$M(r, y, z, t)$" which in the weak field approximation gives the Newtonian results. However, Narlikar [8] pointed out a geometrical defect in the definition of the "*mass function*-$M(r, y, z, t)$". It is well known that the Riemann curvature tensor can be decomposed into Weyl tensor (representing the free

Dedicated to my teacher Professor J. Krishna Rao, whose continuous encouragement motivated me for research.

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gravitational field), the Ricci tensor and its spur (representing the matter fields). Thus the *mass function* must consist of contributions from the gravitational field as well as the material distribution filling it. The "*mass function*-$M(r, y, z, t)$" mentioned above lacks this type of combination as it is related with material density only. To resolve the defects mentioned above Krishna Rao (1971, 1972, 1973) initiated a new formalism, which is more geometrical in nature. In this formalism an eigen value of the Weyl conformal tensor ϵ was introduced in the transverse components of the Einstein tensor, this simplifies mathematical procedure and the "*mass function*-$M(r, t)$" is given by

$$M(r, t) = (4\pi/3)\,(\rho + \epsilon)R^3, \tag{1}$$

where ρ is the proper energy density and R is the Eulerian (or position) coordinate so that $R(r, 0) = r$. The expression for the "*mass function*-$M(r, t)$" given in (1) was obtained an algebraic combination of field equations.

The above formalism has been extended to axially symmetric space-times by Krishna Rao and Seshaiah [4] as well as Dingle's space-times by Krishna Rao and Hasmani [2, 3].

It is well known that the equations of motion for a perfect fluid distribution [10] contain $(\rho + p)$ as the inertial mass per unit volume, where p denotes the isotropic pressure. However, from (1), we note that the Ricci tensor contribution to the *mass function* is in the form of material energy density ρ only and the isotropic pressure p does not contribute in any way. In more general situations where there is an anisotropy on the pressure (1) is modified as

$$M(r, t) = (4\pi/3)\,(\rho + p_\perp - p_r + \epsilon)R^3, \tag{2}$$

where p_r and $p_\perp$ are pressures along radial and transverse directions respectively. From (2) the contribution of pressure anisotropy in the *mass function* is clear. In the case of Dingle's space-times there is complete anisotropy since we can choose pressure in all three space directions as totally different, this case was discussed in [3]. In the present paper we further extend our earlier formalism to obtain general dynamical equations for Dingle's space-times filled with a charged non-perfect fluid and thereby deduce the formula for *mass function*, which contains all pressures and electromagnetic field energy.

In Sect. 2 we introduce the metric and field equations whereas Sect. 3 gives the general dynamical equations. The paper ends with concluding remarks in Sect. 4.

2 The Metric and Field Equations

The metric for Dingle's space-time is given by

$$ds^2 = \exp(\nu)dt^2 - \exp(\lambda)dr^2 - \exp(2\alpha)dy^2 - \exp(2\beta)dz^2, \tag{3}$$

where ν, λ, α and β are functions of all four space-time variables r, y, z and t. The metric (3) does not possess any symmetry properties.

We define the directional derivatives associated with the metric (3) as,

$$D_r = \exp(-\lambda/2)(\partial/\partial r), \tag{4}$$

$$D_y = \exp(-\alpha)(\partial/\partial y), \tag{5}$$

$$D_z = \exp(-\beta)(\partial/\partial z), \tag{6}$$

$$D_t = \exp(-\nu/2)(\partial/\partial t). \tag{7}$$

Assuming that the space-time is filled with a charged non-perfect fluid of proper material density ρ and pressures p_1, p_2, p_3 respectively along r, y, z directions, we write the energy-momentum tensor as [6]

$$T_a^b = \rho u_a u^b + \sum_{i=1}^{3} \left(p_{(i)} v_a^{(i)} v^{b(i)}\right) - F_{ab}F^{bc} + (1/4)\delta_a^b F_{mn}F^{mn}, \tag{8}$$

where

$$v^{(1)a} = (\exp(-\lambda/2), 0, 0, 0), \tag{9}$$

$$v^{(2)a} = (0, \exp(-\alpha), 0, 0), \tag{10}$$

$$v^{(3)a} = (0, 0, \exp(-\beta), 0), \tag{11}$$

$$u^a = (0, 0, 0, \exp(-\nu/2)), \tag{12}$$

and F^{ab} denotes Maxwell's electromagnetic field tensor satisfying Maxwell's equations in the form,

$$F_{[ab,c]} = 0, \qquad \left(\sqrt{-g}F^{ab}\right)_{,b} = \sqrt{-g}J^a, \tag{13}$$

where a square bracket in (13) denotes cyclic rotation of indices within it.

The electromagnetic field energy density $\mathcal{E}$ is given in terms of Maxwell's tensor F^{ik} and its dual $^*F^{ik}$ as

$$\mathcal{E} = (I_1^2 + I_2^2)^{1/2}, \tag{14}$$

where $I_1 = (-1/2)F^{ik}F_{ik}$ and $I_2 = (1/4)^*F_{ik}F^{ik}$.

We now define the invariant velocities along y and z directions as

$$\begin{aligned} U &= u^a\left(\frac{\partial \exp\alpha}{\partial x^a}\right) = D_t(\exp\alpha) = \exp\{\alpha - (\nu/2)\}\dot{\alpha}, \\ V &= u^a\left(\frac{\partial \exp\beta}{\partial x^a}\right) = D_t(\exp\beta) = \exp\{\beta - (\nu/2)\}\dot{\beta}. \end{aligned} \tag{15}$$

Throughout this paper we denote partial differentiation with respect to t and r respectively by an overhead dot and a prime whereas a similar differentiation with respect to y and z coordinates is denoted by suffixes 2 and 3 respectively.

We also define two more quantities as below

$$\begin{aligned} \Gamma &= D_r(\exp\alpha) = \exp\{\alpha - (\lambda/2)\}\alpha', \\ \Sigma &= D_r(\exp\beta) = \exp\{\beta - (\lambda/2)\}\beta'. \end{aligned} \tag{16}$$

The field equations for (3) are written using Dingle's formulae [11] with appropriate modifications after using (8)–(12). Thus,

$$\begin{aligned} 8\pi T_1^1 &= 8\pi(p_1 - \mathcal{E}) = G_1^1 \\ &= \exp(-\lambda)\{\alpha'\beta' + (1/2)\eta'\nu'\} \\ &\quad - \exp(-2\alpha)\{\beta_{22} + (1/2)\nu_{22} + (1/4)\nu_2^2 - (\beta_2 + (1/2)\nu_2)(\alpha_2 - \beta_2)\} \\ &\quad - \exp(-2\beta)\{\alpha_{33} + (1/2)\nu_{33} + (1/4)\nu_3^2 - (\alpha_3 + (1/2)\nu_3)(\alpha_3 - \beta_3)\} \\ &\quad - \exp(-\nu) - \ddot{\alpha} + \ddot{\beta} + \dot{\beta}^2 + 2\dot{\beta} - (1/2)\dot{\eta}\dot{\nu}, \end{aligned} \tag{17}$$

$$\begin{aligned}4\pi(T_2^2+T_3^3) &= -4\pi(p_2+p_3-2\mathcal{E}) = (1/2)(G_2^2+G_3^3)\\ &= 8\pi\epsilon - \exp(-\lambda)\left\{\alpha''+\beta''-\alpha'\beta'-(1/2)\eta'(\lambda'-\nu')\right\}\\ &\quad + \exp(-2\alpha)\big\{\beta_{22}-(1/2)(\lambda_{22}+\nu_{22}+\beta_2^2)-(1/4)(\lambda_2^2+\nu_2^2)\\ &\quad + (\lambda_2+\nu_2)(\alpha_2-\beta_2)-\alpha_2\beta_2\big\}\\ &\quad + \exp(-2\beta)\big\{\alpha_{33}-(1/2)(\lambda_{33}+\nu_{33}+\alpha_3^2-(1/4)(\lambda_3^2+\nu_3^2)\\ &\quad + (\lambda_3+\nu_3)(\alpha_3-\beta_3)-\alpha_3\beta_3\big\}\\ &\quad + \exp(-\nu)\left\{\ddot{\alpha}+\ddot{\beta}+\dot{\alpha}^2+\dot{\beta}^2-\dot{\alpha}\dot{\beta}-(1/2)\dot{\eta}(\dot{\lambda}-\dot{\nu})\right\}, \end{aligned} \tag{18}$$

$$\begin{aligned}8\pi T_4^4 &= 8\pi(\rho+\mathcal{E}) = G_4^4\\ &= -\exp(-\lambda)\left\{\alpha''+\beta''-\alpha'^2+\beta'^2-(1/2)\eta'\lambda'+\alpha'\beta'\right\}\\ &\quad - \exp(-2\alpha)\left(\frac{\lambda_{22}}{2}+\beta_{22}+\frac{\lambda_2^2}{4}+\beta_2^2-\frac{\lambda_2}{2}(\alpha_2-\beta_2)-\alpha_2\beta_2\right)\\ &\quad - \exp(-2\beta)\left\{\frac{\lambda_{33}}{2}+\alpha_{33}+\frac{\lambda_3^2}{4}+\beta_3^2-\frac{\lambda_3}{2}(\alpha_3-\beta_3)-\alpha_3\beta_3\right\}\\ &\quad + \exp(-\nu)\left\{\ddot{\alpha}+\ddot{\beta}+\dot{\alpha}^2+\dot{\beta}^2-\dot{\alpha}\dot{\beta}-(1/2)\dot{\eta}(\dot{\lambda}-\dot{\nu})\right\}, \end{aligned} \tag{19}$$

$$\begin{aligned}8\pi T_2^1 = 0 = G_2^1 &= \exp(-\lambda)\left\{\beta_2'+(1/2)\nu_2'+\beta'\beta_2+(1/4)\nu'\nu_2\right.\\ &\quad \left. -(1/4)\lambda_2(2\beta'+\nu')-(1/2)\alpha'(2\beta_2+\nu_2)\right\}, \end{aligned} \tag{20}$$

$$\begin{aligned}8\pi T_3^1 = 0 = G_3^1 &= \exp(-\lambda)\left\{\alpha_3'+(1/2)\nu_3'+\alpha'\alpha_3+(1/4)\nu'\nu_3\right.\\ &\quad \left. -(1/4)\lambda_3(2\alpha'+\nu')-(1/2)\beta'(2\alpha_3+\nu_3)\right\}, \end{aligned} \tag{21}$$

$$8\pi T_4^1 = 0 = G_4^1 = \exp(-\lambda)\left\{\dot{\alpha}'+\dot{\beta}'+\dot{\alpha}+\dot{\beta}-(1/2)\dot{\lambda}\eta'-(1/2)\nu'\dot{\eta}\right\}, \tag{22}$$

$$\begin{aligned}8\pi T_2^4 = 0 = G_2^4 &= \exp(-\nu)\left\{(1/2)\dot{\lambda}_2+\dot{\beta}_2+(1/4)\lambda_2\dot{\lambda}+\beta_2\dot{\beta}\right.\\ &\quad \left. -(\dot{\alpha}/2)(2\beta_2+\lambda_2)-(\nu_2/4)(2\dot{\beta}+\dot{\lambda})\right\}, \end{aligned} \tag{23}$$

$$\begin{aligned}8\pi T_3^4 = 0 = G_3^4 &= \exp(-\nu)\left\{(1/2)\dot{\lambda}_3+\dot{\alpha}_3+(1/4)\lambda_3\dot{\lambda}+\alpha_3\dot{\beta}\right.\\ &\quad \left. -(\dot{\beta}/2)(2\alpha_3+\lambda_3)-(\nu_3/4)(2\dot{\alpha}+\dot{\lambda})\right\}, \end{aligned} \tag{24}$$

$$\begin{aligned}8\pi T_3^2 = 0 = G_3^2 &= \exp(-2\alpha)\big\{\lambda_{23}+\nu_{23}+(1/2)(\lambda_2\lambda_3+\nu_2\nu_3)\\ &\quad -\alpha_3(\lambda_2+\nu_2)-\beta_2(\lambda_3+\nu_3)\big\}, \end{aligned} \tag{25}$$

where $\eta = \alpha + \beta$ and we have introduced in (18) the Newman-Penrose (1962) scalar Ψ_2 such that

$$\begin{aligned}8\pi\epsilon = -6\Psi_2(1/4)\Big[&\exp(-\lambda)\left\{2(\alpha''+\beta''-\nu''+\alpha'^2+\beta'^2-\nu'^2)-(\lambda'-\nu')\eta'-4\alpha'\beta'\right\}\\ &+\exp(-2\alpha)\big\{\lambda_{22}+\nu_{22}-4\beta_{22}+(1/2)(\lambda_2^2+\nu_2^2)-4\beta_2^2\\ &-(\lambda_2+\nu_2)(\alpha_2-\beta_2)-\lambda_2\nu_2+4\alpha_2\beta_2\big\}\\ &+\exp(-2\beta)\big\{\lambda_{33}+\nu_{33}-4\alpha_{33}+(1/2)(\lambda_3^2+\nu_3^2)-4\alpha_3^2\end{aligned}$$

$$+ (\lambda_3 + \nu_3)(\alpha_3 - \beta_3) - \lambda_3\nu_3 + 4\alpha_3\beta_3\big\}$$
$$+ \exp(-\nu)\big\{2(\ddot{\lambda} - \ddot{\alpha} - \ddot{\beta} + \dot{\lambda}^2 - \dot{\alpha}^2 - \dot{\beta}^2) - (\dot{\lambda} - \dot{\nu})\dot{\eta} - \dot{\lambda}\dot{\nu} + 4\dot{\alpha}\dot{\beta}\big\}\Big]. \tag{26}$$

As earlier [2, 3], we interpret ϵ as the energy density of the free gravitational field.

3 The General Dynamical Equations

The expressions for $\exp(\lambda)$ can be obtained by the algebraic combination of (17) + (19) − (18), thus we get

$$\exp(-\lambda) = \left[UV + K\exp(\eta) - (8\pi/3)\left\{\rho + \epsilon + \frac{p_2 + p_3}{2} - p_1 + 3\mathcal{E}\right\}\exp(\eta)\right]^{-1} \times \exp(\eta)\alpha'\beta' \tag{27}$$

or

$$\exp(-\lambda) = (\Gamma\Sigma)^{-1}\exp(\eta)\alpha'\beta', \tag{28}$$

where

$$\Gamma\Sigma = UV + K\exp(\eta) - (8\pi/3)\left\{\epsilon + \rho + \frac{p_2 + p_3}{2} - p_1 + 3\mathcal{E}\right\}\exp(\eta) \tag{29}$$

and

$$K = \big\{\exp(-2\alpha)(\beta_{22} + \beta_2^2 - \alpha_2\beta_2) + \exp(-2\beta)(\alpha_{33} + \alpha_3^2 - \alpha_3\beta_3\big\} \tag{30}$$

(denotes the Gaussian curvature of the 2-surfaces spanned by the y and z coordinates).

It is also possible to relate $\{\epsilon + \rho + \frac{p_2+p_3}{2} - p_1 + 3\mathcal{E}\}$ with the Gaussian curvature of the 2-surfaces spanned by r, t coordinates (see Robinson and Trautman [9]). In analogy with earlier cases [2, 3] we may write,

$$M(r, y, z, t) = \left\{\epsilon + \rho + \frac{p_2 + p_3}{2} - p_1 + 3\mathcal{E}\right\}\exp(3\eta/2). \tag{31}$$

From this it is clear that the *mass function-*$M(r, y, z, t)$ here consists of the Ricci tensor contribution $\{\rho + \frac{p_2+p_3}{2} - p_1 + 3\mathcal{E}\}$ as well as the Weyl conformal tensor part ϵ.

Using (22) and (27) we obtain

$$\exp(\beta)D_tU + \exp(\alpha)D_tV = (\Gamma\Sigma/2)\{\exp(\alpha)\{\partial\nu/\partial(\exp\alpha)\} + \exp(\beta)\{\partial\nu/\partial(\exp\beta)\}\} + \exp(\eta)\left\{X - (8\pi/3)\left(\epsilon + \rho + \frac{p_2 + p_3}{2} + 2p_1\right)\right\}, \tag{32}$$

where

$$X = \exp(-\nu/2)\left[\left\{D_{yy} + D_{zz} + D_y^2 + D_z^2\right\}\exp(\nu/2)\right]. \tag{33}$$

The baryon conservation equation $(nu^a)_{;a} = 0$ simplifies to

$$D_t(n\exp(\eta)) = -n\exp(\eta)\left[\exp(\beta)\left\{\frac{\partial U}{\partial(\exp\eta)}\right\} + \exp(\alpha)\left\{\frac{\partial V}{\partial(\exp\eta)}\right\}\right]. \tag{34}$$

Applying D_t to (27) and then using (17), (22) we get

$$\begin{aligned} D_t M &= (1/2)\exp(\eta/2) D_t(K\exp\eta) + (1/4)\exp(\eta/2) W(X - 8\pi p_1) \\ &\quad - (1/4)\exp(\eta/2)\big[D_t(UV + \Sigma\Gamma) - (1/2)\exp(\eta)\big(\exp(-2\alpha)D_t(U^2+\Gamma^2) \\ &\quad + \big(\exp(-2\beta)D_t(V^2+\Sigma^2)\big)\big)\big], \end{aligned} \tag{35}$$

where

$$W = \exp(\nu/2)\{\exp(\beta)U + \exp(\alpha)V\} = \exp(\eta)\dot{\eta}. \tag{36}$$

We may term (13), (14), (30), (31) and (35) as "*dynamical equations*" since they govern evolution of the system.

Similarly, we get

$$\begin{aligned} D_r M &= (1/2)\exp(\eta/2) D_r(K\exp\eta) + (1/4)\exp(\eta/2) Z(Y - 8\pi\rho) \\ &\quad - (1/4)\exp(\eta/2)\big[D_r(UV + \Sigma\Gamma) - (1/2)\exp(\eta)\big(\exp(-2\alpha)D_r(U^2+\Gamma^2) \\ &\quad + (1/4)\exp(\eta/2)\,\big(\exp(-2\beta)D_r(V^2+\Sigma^2)\big)\big)\big], \end{aligned} \tag{37}$$

where

$$Z = \exp(\lambda/2)\{\exp(\beta)\Gamma + \exp(\alpha)\Sigma\}, \tag{38}$$

$$Y = \exp(-\lambda/2)\left[\left\{D_{yy} + D_{zz} + D_y^2 + D_z^2\right\}\exp(\lambda/2)\right] \tag{39}$$

we further obtain,

$$\begin{aligned} &\big(\exp(\alpha)D_y + \exp(\beta)D_z\big)M \\ &\quad = \big(\exp(\alpha)D_y + \exp(\beta)D_z\big)\{K\exp(3\eta/2)\} + \exp(\eta/2)[UD_t\{(\exp\beta)_2 + (\exp\alpha)_3\} \\ &\qquad + VD_t\{(\exp\alpha)_2 + (\exp\beta)_3\}] + \exp(\eta/2)[UV\{\eta_2 + \eta_3 - 2(\nu_2+\nu_3)\} \\ &\qquad - \Gamma\Sigma\{\eta_2 + \eta_3 - 2(\lambda_2+\lambda_3)\} - \exp(\eta/2)[\Gamma D_r\{(\exp\beta)_2 + (\exp\beta)_3\} \\ &\qquad + \Sigma D_r\{(\exp\alpha)_2 + (\exp\alpha)_3\}]. \end{aligned} \tag{40}$$

4 Conclusion

We have derived general dynamical equations for the Dingle's space-times filled with charged non-perfect fluid, as pointed out earlier these equations govern the evolution of the system. Also the expression of the *mass function*-$M(r, y, z, t)$ is obtained for this general situation and the contributions of pressure anisotropy, electromagnetic field energy and energy density of free gravitational field are made explicit. We hope that the present formalism will help to analyze the evolution of physical systems of general nature by taking into account the different contributions to the mass function mentioned above.

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